ON THE RADIATIVE EQUILIBRIUM OF A PLANETARY NEBULA[1]

The excellent works of Hubble, Bowen and Zanstra have solved at least qualitatively the problem of the origin of nebular luminosity and of nebular spectra. Carroll and Cillié have made an attempt to compute the relative intensities of the members of the Balmer series of hydrogen in nebular spectra, opening the door to theoretical interpretation of modern spectrophotometric data. However, the dynamics of nebulae as well as the nature and the origin of forces acting in them remain unknown. It is now scarcely possible to answer these questions and to form a complete theory of planetary nebulae.

Selective radiation pressure, owing to specific nebular conditions, plays a very important part in the nebulae. Perhaps radiation pressure is in this case greater than any other force. Computation of radiation pressure is therefore a matter of considerable interest. This computation can be based on the analysis of the field of radiation. An approximate analysis can be done without the solution of other problems connected with planetary nebulae. The purpose of the present paper is to investigate the radiative equilibrium of planetary nebulae. We concentrate on planetary nebulae, though many results may be applied to the diffuse nebulae, as well as to the gaseous envelopes surrounding some stars with emission line spectrum (P Cygni and others). The method applied here was proposed by the author in an earlier paper [1].

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The Radiative Equilibrium of a Non-expanding Hydrogen Nebula

We will consider below planetary nebula consisting of hydrogen only. There are some observational data indicating the expansion of planetary nebulae. (Owing to Doppler displacement, the frequencies of a spectral line in different parts of a nebula are different.) This yields an appreciable change in the type of radiative equilibrium.

But in some cases the expansion velocity is so small that the frequency differences of a line for different parts of the nebula are smaller than its Doppler-broadening, caused by the thermal motion of atoms. The behavior of such nebulae is the same as the behavior of a non-expanding nebula, if we confine ourselves to the field of radiation and its interaction with nebular matter.

According to the theory of nebular luminosity developed by Zanstra, all, or at least a considerable part, of the quanta emitted by the central star, which have frequencies greater than ν_0 (the frequency of the limit of the Lyman series), are absorbed by the hydrogen atoms in the nebula. This circumstance requires that the optical thickness of the nebula τ_1 for these frequencies should be larger than unity or at least not much smaller than unity.

Following Zanstra's line of argument, we will show that in the place of each absorbed quantum having a frequency greater than ν_0 there is a certain probability p of re-emission in the same frequency and the probability $1-p$ of re-emission in the line $\mathbf{L}_\alpha$ (first line of the Lyman series of hydrogen). For simplicity, we will call the radiation beyond the head of the Lyman series briefly "ultraviolet radiation" and the corresponding quanta "ultraviolet quanta."

We consider the possible transformations of an ultraviolet quantum, which is emitted from the surface of the central star. It will be absorbed by the nebular envelope, and the absorption will be accompanied by the ionization of a hydrogen atom. After an interval of time the electron that became free will be captured again by a proton. There are two possibilities at this capture: (I) the electron will jump immediately into the deepest level $1S$ (first level); and (II) the electron will jump into one of the excited levels.

In Case I, a new ultraviolet quantum is emitted and the initial state is restored. In Case II, the electron makes a chain of transitions, the last link of which will be a transition into the first level. The dilution of radiation is so great, and the density of matter so low, that interruption of these transitions is very improbable. The last transition into the normal state is accompanied by the emission of a quantum of the Lyman series. There are again two possibilities:

(a) An $\mathbf{L}_\alpha$-quantum is emitted. However, Zanstra's theory requires that the optical thickness of the nebula in ultraviolet light be at least of the order of unity. But the coefficient of line absorption in the Lyman series is some thousand times larger than the absorption coefficient beyond the head of the series. The emitted $\mathbf{L}_\alpha$-quantum, therefore, will be absorbed by a hydrogen atom in the normal state. This atom passes into the second level, and then, owing to the absence of external perturbations during its short lifetime, turns back to the first level, emitting again an $\mathbf{L}_\alpha$-quantum. Thus the $\mathbf{L}_\alpha$-quantum remains unchanged, and we may say that it is merely scattered. These processes may be repeated many times until the quantum reaches the outer boundary of the nebula and escapes.

(b) A quantum of some other line of the Lyman series is emitted. For simplicity we assume that it is an $\mathbf{L}_\beta$-quantum. In this line the optical thickness of the nebula is also very large, and the emitted quantum will be absorbed. This absorption is accompanied by the transition of an atom from the first to the third level. The atom in the third level has two possibilities: it either makes the transition of the type $3 \longrightarrow 2 \longrightarrow 1$, emitting the quanta $\mathbf{H}_\alpha$ and $\mathbf{L}_\alpha$ successively, or it passes immediately into the first level, emitting again $\mathbf{L}_\beta$. In the first case the final product is an $\mathbf{L}_\alpha$-quantum. Its further fate is described above. In the second case the quantum $\mathbf{L}_\beta$ will be absorbed, and thus there exists a finite probability of creation of a quantum $\mathbf{L}_\alpha$. After many absorptions and re-emissions the probability of the survival of an $\mathbf{L}_\beta$-quantum will be very small and the probability of the creation of an $\mathbf{L}_\alpha$-quantum will be practically equal to unity.

In this manner in both cases (a) and (b) the final product is an $\mathbf{L}_\alpha$-quantum. It is easily seen that our considerations may be generalized to the cases where the transitions are accompanied, instead of by the emission of an $\mathbf{L}_\beta$-quantum, by the emission of an $\mathbf{L}_\gamma$, $\mathbf{L}_\delta$, etc. Let p be the probability of Case I and $1-p$ the probability of Case II.

Thus we see indeed that after the absorption of an ultraviolet quantum there is a finite probability p of re-emission of it with the same wavelength[2] and a finite probability $1-p$ of re-emission of quantum $\mathbf{L}_\alpha$. We shall not take into account the intermediate stages in which the absorbed quantum may appear as $\mathbf{L}_\beta$-, $\mathbf{L}_\gamma$-quantum, etc. These will have little influence on the results. The quanta $\mathbf{L}_\alpha$ cannot be transformed and can only be scattered.

In this way the problem is reduced to the study of two superposed fields of radiation: the field of ultraviolet quanta and the $\mathbf{L}_\alpha$-radiation field in the planetary nebulae.

The Field of Ultraviolet Quanta. In this paper we shall use the method of reduction of a spherical problem to a plane problem, developed by Professor Milne. Let k be the absorption coefficient of the ultraviolet radiation per atom. This coefficient depends upon the wavelength. We shall use its mean value. Let, further, n be the number of H atoms in the first level in $1\mathrm{cm}^3$, r_1 and r_2 the distances of the inner and outer boundary of the nebular ring from the central star. Then the optical depth at the distance r from the central star is

$$\tau = \int_{r_1}^{r_2} nk\,dr. \tag{1}$$

The equations of transfer of the energy of ultraviolet quanta may be written in Milne form

$$\frac{1}{2}\frac{dI(\tau)}{d\tau} = I(\tau) - B(\tau) \tag{2}$$

[2]Owing to the free-free transitions as well as to the inelastic collisions of free electrons the wavelength of the re-emitted quantum may differ considerably from the wavelength of the absorbed quantum. But it always remains shorter than $\lambda_0 = \frac{c}{\nu_0}$, where c is the velocity of light. In this paper we do not distinguish between the ultraviolet quanta of different wavelengths.

$$\frac{1}{2}\frac{dI'(\tau)}{d\tau} = B(\tau) - I'(\tau) \tag{3}$$

if we use an approximation of Schwarzschild-Schuster type.

Here $I(\tau)$ is the average intensity of the diffuse ultraviolet radiation of the nebula in the outward direction at point τ, and $I'(\tau)$ is the average intensity of the same radiation at the same point in the inward direction. The quantity $4\pi\, B(\tau)\, d\tau$ is the amount of energy of ultraviolet quanta emitted in the layer $d\tau$ per second. The same layer absorbs the diffuse ultraviolet radiation from various parts of the nebular ring. The absorbed energy is equal to $2\pi[I(\tau) + I'(\tau)]\, d\tau$. Besides this, the layer absorbs the radiation of the central star. Let πS be the amount of ultraviolet energy falling on each square centimeter of the inner surface of the nebula. At the point τ this amount is reduced to $\pi S\, \exp\left(-(\tau_1 - \tau)\right)$, where

$$\tau_1 \equiv \int_{r_1}^{r_2} nk\, dr \tag{4}$$

is the optical thickness of the nebula. From this amount our layer absorbs

$$\pi S\, \exp\left(-(\tau_1 - \tau)\right) d\tau.$$

Since from the quanta absorbed only the portion p is re-emitted again as ultraviolet quanta, the equation of radiative equilibrium may be written:

$$p\left[I(\tau) + I'(\tau) + \frac{1}{2} \cdot S\, \exp\left(-(\tau_1 - \tau)\right)\right] = 2\, B(\tau). \tag{5}$$

Introducing the boundary conditions [2]

$$I'(0) = 0, \quad I(\tau_1) = I'(\tau_1), \tag{6}$$

we take into account the diffuse radiation incident on any portion of the inner face of the nebular envelope and arriving from other portions of the inner face.

From equations (2) and (3) we have

$$\frac{1}{2} \cdot \frac{d(I + I')}{d\tau} = I - I' \tag{7}$$

$$\frac{1}{2} \cdot \frac{d(I - I')}{d\tau} = I + I' - 2B. \tag{8}$$

Differentiating (7) and comparing with (8) we obtain

$$\frac{1}{4} \cdot \frac{d^2(I + I')}{d\tau^2} = I + I' - 2B. \tag{9}$$

Substituting (5) into (9) we find the following equation for $I + I'$:

$$\frac{1}{4} \cdot \frac{d^2(I + I')}{d\tau^2} = (1 - p)(I + I') - \frac{p}{2} \cdot S \exp(-(\tau_1 - \tau)). \tag{10}$$

The general solution of this equation is

$$I + I' = A \exp(-\lambda\tau) + B \exp(\lambda\tau) + \frac{2p}{3 - 4p} S \exp(-(\tau_1 - \tau)), \tag{11}$$

where A and B are constants of integration and $\lambda = 2\sqrt{1 - p}$. Substituting (11) in (5) we find

$$B(\tau) = \frac{p}{a}\left(A \exp(-\lambda\tau) + B \exp(\lambda\tau) + \frac{3}{2(3 - 4p)} \cdot S \exp(-(\tau_1 - \tau))\right). \tag{12}$$

Substituting (11) in (7) we obtain

$$I(\tau) - I'(\tau) = -\frac{\lambda}{2} A \exp(-\lambda\tau) + \frac{\lambda}{2} B \exp(\lambda\tau) + \frac{p}{3 - 4p} \cdot S \exp(-(\tau_1 - \tau)). \tag{13}$$

Adding and subtracting (11) and (13), we find $I(\tau)$ and $I'(\tau)$:

$$I(\tau) = \frac{1}{2}\left(1 - \frac{\lambda}{2}\right) A \exp(-\lambda\tau) + \frac{1}{2}\left(1 + \frac{\lambda}{2}\right) B \exp(\lambda\tau) + \frac{3p}{2(3 - 4p)} \cdot S \exp(-(\tau_1 - \tau)), \tag{14}$$

$$I'(\tau) = \frac{1}{2}\left(1 + \frac{\lambda}{2}\right) A \exp(-\lambda\tau) + \frac{1}{2}\left(1 - \frac{\lambda}{2}\right) B \exp(\lambda\tau) + \frac{p}{2(3 - 4p)} \cdot S \exp(-(\tau_1 - \tau)). \tag{15}$$

The first of conditions (6) may be written according to (15) in the form:

$$A\left(1+\frac{\lambda}{2}\right)+B\left(1-\frac{\lambda}{2}\right)+\frac{p}{3-4p}\cdot S\,\exp\left(-\tau_1\right)=0. \tag{16}$$

The second of conditions (6) gives

$$\lambda B\exp\left(\lambda\tau_1\right)+\frac{2pS}{3-4p}=\lambda A\cdot\exp\left(-\lambda\tau_1\right). \tag{17}$$

From equations (16) and (17) we find the following coefficients A and B:

$$A=\frac{\left(1-\frac{\lambda}{2}\right)\exp\left(\tau_1\right)-\frac{\lambda}{2}\exp\left(\lambda\tau_1\right)}{\frac{\lambda}{2}\left[\left(1-\frac{\lambda}{2}\right)\exp\left(-\lambda\tau_1\right)+\left(1+\frac{\lambda}{2}\right)\exp\left(\lambda\tau_1\right)\right]}\cdot\frac{p\cdot S\,\exp\left(-\tau_1\right)}{3-4p}, \tag{18}$$

$$B=\frac{\left(1+\frac{\lambda}{2}\right)\exp\left(\tau_1\right)+\frac{\lambda}{2}\exp\left(-\lambda\tau_1\right)}{\frac{\lambda}{2}\left[\left(1-\frac{\lambda}{2}\right)\exp\left(-\lambda\tau_1\right)+\left(1+\frac{\lambda}{2}\right)\exp\left(\lambda\tau_1\right)\right]}\cdot\frac{p\cdot S\,\exp\left(-\tau_1\right)}{3-4p}. \tag{19}$$

These values of A and B in conjunction with (12), (14) and (15) give the solution for the field of the ultraviolet quanta.

For the net flux of the diffuse ultraviolet radiation at the outer boundary of the nebula we obtain:

$$\pi F_u=\pi\left[I(0)-I'(0)\right]=$$
$$\pi\left[1-\frac{2\exp\left(\tau_1\right)-\lambda\sinh\left(\lambda\tau_1\right)}{\left(1-\frac{\lambda}{2}\right)\exp\left(\lambda\tau_1\right)+\left(1+\frac{\lambda}{2}\right)\exp\left(\lambda\tau_1\right)}\right]\cdot\frac{p\cdot S\,\exp\left(-\tau_1\right)}{3-4p}. \tag{20}$$

For the representation of the solution in numerical form it is necessary to know p and τ_1. The value of p can be calculated from pure physics. Cillié [3] has computed the relative probabilities of the capture of electrons by protons on different hydrogen levels. From his results we have deduced the fraction of captured electrons which pass immediately from a free state to the first level, re-emitting the ultraviolet quanta. This fraction is our p. The value of p depends on the temperature of free electrons. For different temperatures we have:

T	$10,000^\circ$	$20,000^\circ$	$50,000^\circ$
p	0.46	0.49	0.57

Putting $p = 0.5$, we find for large values of τ_1 ($\tau_1 > 3$) the following asymptotic expression for the net flux πF_u at the outer boundary:

$$\pi F_u = 0.7\,\pi S\, e^{-\tau_1}.$$

The net flux of the direct ultraviolet radiation of a star will be simply $\pi S e^{-\tau_1}$, while the whole net flux will be $1.7\,\pi S e^{-\tau_1}$.

In the absence of the absorbing envelope the net flux from the star is πS. The fraction $1 - 1.7\,e^{-\tau_1}$ of it is converted into other forms of radiation. Owing to the fact that by the splitting of an ultraviolet quantum an $\mathbf{L}_\alpha$-quantum is certainly created, the net flux in the line $\mathbf{L}_\alpha$ at the outer boundary will contain $\dfrac{1 - 1.7\,e^{-\tau_1}}{h\,\nu_c}$ $\mathbf{L}_\alpha$-quanta, where ν_c is the average frequency of the ultraviolet quanta. If τ_1 is large, the flux of the $\mathbf{L}_\alpha$ energy at the outer boundary of nebula is approximately $\dfrac{\nu_\alpha}{\nu_c}\,\pi S$. Here ν_α is the frequency of the line $\mathbf{L}_\alpha$.

The Field of $\mathbf{L}_\alpha$-Radiation. Let κ be the absorption coefficient within the line $\mathbf{L}_\alpha$ per hydrogen atom in the normal state. The optical depth for this line is defined by

$$t = \int_r^{r_2} n\kappa\, dr. \tag{21}$$

The ratio $\dfrac{\kappa}{k} = \omega$ may be assumed constant when the temperature variations within the nebula are neglected. In fact, κ is a function of atomic constants and of the Doppler breadth of the line only. This breadth depends upon the temperature. When $\dfrac{\kappa}{k} = \omega$ is constant, the ratio $\dfrac{t}{\tau}$ is also constant, and we have

$$\frac{t}{\tau} = \frac{\kappa}{k} = \omega. \tag{22}$$

If the temperature of the nebula is of the order $10^3 - 10^4$ degrees, the quantity ω will also be of the order $10^3 - 10^4$. Since we have supposed that the optical thickness τ_1 of the nebula in the ultraviolet region is of the order of unity or larger, the optical thickness in the line $\mathbf{L}_\alpha$,

$$t_1 = \int_{r_1}^{r_2} n\kappa\, dr,$$

will be of the order $10^3 - 10^4$, or larger.

The equation of transfer of the radiation in the line $\mathbf{L}_\alpha$ has the same form as (2) and (3). Let $K(t)$ be the average intensity of the diffuse $\mathbf{L}_\alpha$-radiation of the nebula in the outward direction at point t, and $K'(t)$ be the average intensity of the same radiation at the same point in the inward direction. The equations of transfer are:

$$\frac{1}{2}\frac{dK(t)}{dt} = K(t) - C(t), \tag{23}$$

$$\frac{1}{2}\frac{dK'(t)}{dt} = C(t) - K'(t), \tag{24}$$

where $4\pi C(t)\,dt$ is the amount of energy emitted by the layer dt in the line $\mathbf{L}_\alpha$ per second. This layer absorbs the diffuse $\mathbf{L}_\alpha$-radiation from the other parts of the nebula. The quantity of diffuse radiation absorbed is $2\pi\,[K(t) + K'(t)]\,dt$. The number of $\mathbf{L}_\alpha$-quanta emitted by the central star is negligible, since the number of ultraviolet quanta transformed into $\mathbf{L}_\alpha$-quanta is some thousand times larger.

The number of ultraviolet quanta which are absorbed in the layer dt and are transformed into $\mathbf{L}_\alpha$-quanta is

$$\frac{(1-p)[2\pi(I + I') + \pi\,S\exp\left(-(\tau_1 - \tau)\right)]\,dt}{h\,\nu_c}.$$

Thus the $\mathbf{L}_\alpha$-radiation created in dt according to (5) is

$$\frac{1-p}{p}\cdot\frac{\nu_\alpha}{\nu_c}\cdot 4\pi\,B(\tau)\,d\tau = 4\pi\,\frac{1-p}{p}\cdot\frac{\nu_\alpha}{\nu_c}\cdot B(\tau)\,\frac{dt}{\omega}.$$

Hence the equation of radiative equilibrium is

$$4\pi\,C(t)\,dt = 2\pi\,[K(t) + K'(t)]\,dt + 4\pi\,\frac{1-p}{p}\cdot\frac{\nu_\alpha}{\nu_c}\cdot B(\tau)\,\frac{dt}{\omega},$$

or

$$C(t) = \frac{1}{2}[K(t) + K'(t)] + \frac{\nu_\alpha}{\nu_c}\cdot\frac{1-p}{p\,\omega}\,B(\tau). \tag{25}$$

The boundary conditions are

$$K'(0) = 0, \quad K'(t_1) = K(t_1). \tag{26}$$

From equations (23) and (24) we have

$$\frac{1}{2}\frac{d(K+K')}{dt} = K - K', \tag{27}$$

$$\frac{1}{2}\frac{d(K-K')}{dt} = K + K' - 2C(t). \tag{28}$$

Differentiating (27) and using (28) we find

$$\frac{1}{4}\frac{d^2(K+K')}{dt^2} = K + K' - 2C(t), \tag{29}$$

or according to (25)

$$\frac{1}{4}\frac{d^2(K+K')}{dt^2} = -\frac{2\nu_\alpha}{\nu_c}\cdot\frac{1-p}{p\,\omega}B(\tau). \tag{30}$$

Writing $B(\tau)$ in the form

$$\begin{aligned} B(\tau) &= \frac{p}{2}\left(A\,e^{-\lambda\tau} + Be^{\lambda\tau} + D\exp\left(-(\tau_1-\tau)\right)\right) \\ &= \frac{p}{2}\left(A\,e^{-\frac{\lambda}{\omega}\,t} + B\,e^{\frac{\lambda}{\omega}\,t} + D\,e^{-\frac{t_1-t}{\omega}}\right), \end{aligned} \tag{31}$$

where

$$D = \frac{3}{2(3-4p)}\cdot S, \tag{32}$$

we find the following solution of equation (30):

$$K(t) + K'(t) = a + bt - \frac{4\nu_\alpha}{\nu_c}\cdot\frac{1-p}{\lambda^2}\,\omega\left(A\,e^{-\frac{\lambda}{\omega}\,t} + B\,e^{\frac{\lambda}{\omega}\,t} + D\lambda^2\,e^{-\frac{t_1-t}{\omega}}\right),$$

where a and b are constants of integration.

Differentiating this expression and using (27), we find

$$K(t) - K'(t) = \frac{b}{2} - \frac{2\nu_\alpha}{\nu_c}\cdot\frac{1-p}{\lambda}\left(-A\,e^{-\frac{\lambda}{\omega}\,t} + B\,e^{\frac{\lambda}{\omega}\,t} + D\lambda\,e^{-\frac{t_1-t}{\omega}}\right).$$

According to the definition of λ,

$$\lambda = 2\sqrt{1-p}.$$

Therefore,

$$K(t) + K'(t) = a + bt - \frac{\nu_\alpha}{\nu_c}\,\omega\left(A\,e^{-\frac{\lambda}{\omega}t} + B\,e^{\frac{\lambda}{\omega}t} + D\lambda^2\,e^{-\frac{t_1-t}{\omega}}\right), \qquad (33)$$

$$K(t) - K'(t) = \frac{b}{2} - \frac{\nu_\alpha}{\nu_c}\cdot\frac{\lambda}{2}\left(-A\,e^{-\frac{\lambda}{\omega}t} + B\,e^{\frac{\lambda}{\omega}t} + D\lambda\,e^{-\frac{t_1-t}{\omega}}\right). \qquad (34)$$

From (33) and (34) we have

$$\begin{aligned} K(t) = \frac{a}{2} + \frac{b}{4} + \frac{b}{2}t - \frac{\nu_\alpha}{2\nu_c}\,\omega\Bigg[A\left(1 - \frac{\lambda}{2\omega}\right)e^{-\frac{\lambda}{\omega}t} + \\ B\left(1 + \frac{\lambda}{2\omega}\right)e^{\frac{\lambda}{\omega}t} + D\lambda^2\left(1 + \frac{1}{2\omega}\right)e^{-\frac{t_1-t}{\omega}}\Bigg], \end{aligned} \qquad (35)$$

$$\begin{aligned} K'(t) = \frac{a}{2} - \frac{b}{4} + \frac{b}{2}t - \frac{\nu_\alpha}{2\nu_c}\,\omega\Bigg[A\left(1 + \frac{\lambda}{2\omega}\right)e^{-\frac{\lambda}{\omega}t} + \\ B\left(1 - \frac{\lambda}{2\omega}\right)e^{\frac{\lambda}{\omega}t} + D\lambda^2\left(1 - \frac{1}{2\omega}\right)e^{-\frac{t_1-t}{\omega}}\Bigg]. \end{aligned} \qquad (36)$$

The conditions (26) are reduced to

$$\begin{aligned} &\frac{a}{2} - \frac{b}{4} = \\ &\frac{\nu_\alpha}{2\nu_c}\omega\left[A\left(1 + \frac{\lambda}{2\omega}\right) + B\left(1 - \frac{\lambda}{2\omega}\right) + 4D(1-p)\left(1 - \frac{1}{2\omega}\right)e^{-\frac{t_1}{\omega}}\right], \end{aligned} \qquad (37)$$

and

$$b = \frac{\nu_\alpha}{\nu_c}\left[-\lambda A\,e^{-\lambda\tau_1} + \lambda B\,e^{\lambda\tau_1} + 4D(1-p)\right]. \qquad (38)$$

Substituting (38) in (37) we find:

$$\begin{aligned} a = \frac{\nu_\alpha}{2\nu_c}\left[-\lambda A\,e^{-\lambda\tau_1} + \lambda B\,e^{\lambda\tau_1} + 4D(1-p)\right] + \\ \frac{\nu_\alpha}{\nu_c}\omega\left[A\left(1 + \frac{\lambda}{2\omega}\right) + B\left(1 - \frac{\lambda}{2\omega}\right) + 4D(1-p)\left(1 - \frac{1}{2\omega}\right)e^{-\frac{t_1}{\omega}}\right]. \end{aligned} \qquad (39)$$

Equations (35) and (36) together with (38) and (39) give the solution for the $\mathbf{L}_\alpha$-field.

The Density of Radiation in the Inner Layers of the Nebula. We have denoted above by πS the amount of energy of the ultraviolet radiation falling from the star on each square centimeter of the inner surface of the nebula. In the absence of re-emission the mean intensity of the ultraviolet radiation in this region will equal $\frac{\pi S}{4\pi} = 0.25\,S$. In the case where re-emission is taken into account, the average intensity of ultraviolet radiation increases and is equal to $\frac{1}{4}\,S + \frac{1}{2}(I_1 + I_2)$. According to (11), (18) and (19),

$$I(\tau_1) + I'(\tau_1) = \frac{2p}{3-4p} \cdot S \left[1 - \frac{1}{\lambda} \cdot \frac{\lambda\, e^{-\tau_1} + \lambda \cosh(\lambda\tau_1) + 2\sinh(\lambda\tau_1)}{2\cosh(\lambda\tau_1) + \lambda \sinh(\lambda\tau_1)}\right]. \tag{40}$$

The expression in brackets remains between 0 and $1 - \frac{1}{\lambda}$, when τ_1 changes between 0 and ∞. Therefore we have

$$\frac{1}{4} \cdot S \leq \frac{1}{4} \cdot S + \frac{1}{2}(I_1 + I_2) \leq \frac{1}{4} \cdot S + \frac{2p\left(1 - \frac{1}{\lambda}\right)}{3 - 4p} \cdot S.$$

Putting $p = 0.5$, we obtain:

$$\frac{1}{4} \cdot S \leq \frac{1}{4} \cdot S + \frac{1}{2}(I_1 + I_2) < 0.40\,S.$$

Thus the mean intensity of ultraviolet radiation at the inner boundary of nebula is of the same order of magnitude as in the absence of a nebular envelope. It should be doubled if τ_1 is very large.

The state of affairs entirely changes when we consider the $\mathbf{L}_\alpha$-field. Owing to the large optical thickness of the nebula in the $\mathbf{L}_\alpha$-line, and to the fact that all $\mathbf{L}_\alpha$-quanta absorbed are re-emitted in the same frequency, the density of $\mathbf{L}_\alpha$-radiation in the inner layers of the ring is very large.

In order to estimate this density we consider a modification of (38) and (39). In fact, comparing (38) with (17) and (32), we find

$$b = \frac{2\nu_\alpha}{\nu_c} \cdot S, \tag{41}$$

and neglecting in (39) the terms not containing the factor ω

$$a = \frac{\nu_\alpha}{\nu_c}\,\omega \cdot \left[A + B + 4D(1-p)\,e^{-\tau_1}\right]. \tag{42}$$

Substituting the values of A, B and D, we find:

$$a = \frac{\nu_\alpha}{\nu_c}\,\omega \cdot \left[3(1-p)\,e^{-\tau_1} - \frac{1 + e^{-\tau_1}\cosh(\lambda\tau_1)}{2\cosh(\lambda\tau_1) + \lambda\sinh(\lambda\tau_1)}\right]\frac{2p}{3-4p}\cdot S. \quad (43)$$

For the brackets in (33) we have

$$A\,e^{-\lambda\tau_1} + B\,e^{\lambda\tau_1} + D\,\lambda^2 = \frac{S}{3-4p}\left[6(1-p) - \frac{2p}{\lambda}\cdot\frac{\lambda\,e^{-\tau_1} + \lambda\,\cosh(\lambda\tau_1) + 2\sinh(\lambda\tau_1)}{2\cosh(\lambda\tau_1) + \lambda\sinh(\lambda\tau_1)}\right]. \quad (44)$$

This expression varies within the limits

$$2S \le A\,e^{-\lambda\tau_1} + B\,e^{\lambda\tau_1} + D\,\lambda^2 < \frac{S}{3-4p}\left[6(1-p) - \frac{2p}{\lambda}\right]$$

or, if $p = 0.5$

$$2S \le A\,e^{-\lambda\tau_1} + B\,e^{\lambda\tau_1} + D\,\lambda^2 \le 2.29\,S,$$

and in the first approximation

$$A\,e^{-\lambda\tau_1} + B\,e^{\lambda\tau_1} + D\,\lambda^2 = 2.15\,S. \quad (45)$$

For the mean intensity of $\mathbf{L}_\alpha$-radiation at the inner boundary, we obtain approximately:

$$\frac{1}{2}\left[K(t_1) + K'(t_1)\right] = \frac{\nu_\alpha}{\nu_c}\,\omega\,S \cdot \left[\tau_1 + \frac{1}{2}f(\tau_1) - 1.07\right],$$

where

$$f(\tau_1) = \left[3(1-p)\cdot e^{-\tau_1} - \frac{1 + e^{-\tau_1}\cosh(\lambda\tau_1)}{2\cosh(\lambda\tau_1) + \lambda\sinh(\lambda\tau_1)}\right].$$

If $\tau \ge 2$ we may neglect $\frac{1}{2}f(\tau_1)$ and have approximately

$$\frac{1}{2}\left[K(t) + K'(t)\right] = \frac{\nu_\alpha}{\nu_c}\,\omega\,S \cdot [\tau_1 - 1].$$

We may take $\omega = 10,000$ (see [4]). Therefore, if $\tau_1 = 2$ the mean density of the $\mathbf{L}_\alpha$-radiation at the inner boundary will be of the order $10,000\,\pi\,S$, where πS is again the energy of the whole ultraviolet radiation falling on each square centimeter of the inner surface of the nebula from the central star. Therefore, the density of $\mathbf{L}_\alpha$-radiation in this example is 40,000 times larger than the density of the whole diluted ultraviolet radiation of the nucleus in the absence of the absorbing shell at the same distance. A rough estimate shows that the ultraviolet radiation of the black body at temperatures of the order $40,000^\circ - 50,000^\circ$ is about $5 \cdot 10^4$ times stronger than the same radiation within the Doppler width of the $\mathbf{L}_\alpha$-line corresponding to the temperature of the nebular matter. Thus the density of $\mathbf{L}_\alpha$-radiation in the inner layers of the nebular envelope will be

$$40,000 \times 5 \cdot 10^4 = 2 \cdot 10^9$$

times larger than the density of the direct $\mathbf{L}_\alpha$-radiation of the central star within the Doppler line-width.

Such a large density of $\mathbf{L}_\alpha$-radiation will produce a large accumulation of atoms in the state $2P$. On the other hand there will also be a large accumulation of hydrogen atoms in the metastable state $2S$. It may happen, therefore, that the optical thickness of the nebula in the lines of the Balmer series will not be very small.

Radiation Pressure in the Outer Parts of the Nebula. The greater part of the ultraviolet radiation of the star is transformed by the nebula into $\mathbf{L}_\alpha$-quanta. Therefore, the flux of radiation emitted by the nebula will consist chiefly of $\mathbf{L}_\alpha$-quanta. For sufficiently large τ_1 each ultraviolet quantum will give rise to an $\mathbf{L}_\alpha$-quantum, and the flow of $\mathbf{L}_\alpha$-radiation from the nebula will be of the order $\frac{\nu_\alpha}{\nu_c} \cdot \pi S$. On the inner surface of the nebula the flux of $\mathbf{L}_\alpha$-radiation will be practically equal to zero, and the flux of ultraviolet radiation will be πS. Now the radiation pressure in a layer of gas is proportional to the absorption coefficient. On the inner surface of the nebula the resulting flux of radiation consists of ultraviolet quanta, for which the absorption coefficient is small. Therefore, the radiation pressure will not be very great. In the outer parts of the nebula, on the contrary, the flux of radiation consists chiefly of $\mathbf{L}_\alpha$-quanta, and the absorption coefficient is about 10^4 times larger than in the case of ultraviolet quanta, while the flux of radiation is of the same order. The radiation pressure, or more exactly, the gradient of radiation pressure, will be here 10^4 times greater than on the inner boundary of the ring. It is physically clear that for large τ_1 the net flux of $\mathbf{L}_\alpha$-radiation πF_α in the outer layer of the ring will be determined by

$$\pi F_\alpha = \left(\frac{r}{r_n}\right)^2 \cdot \frac{2\pi h\nu_\alpha}{c^2} \int_{\nu_0}^{\infty} \frac{\nu^2 \, d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1},$$

where r and r_n are respectively the radius of the central star and the radius of the nebula. The average impulse received by a hydrogen atom in a normal state per second from $\mathbf{L}_\alpha$-quanta will be

$$\frac{\kappa\pi F_\alpha}{c} = \left(\frac{r}{r_n}\right)^2 \cdot \frac{2\kappa\pi h\nu_\alpha}{c^3} \int_{\nu_0}^{\infty} \frac{\nu^2 \, d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1}.$$

The impulse received by each hydrogen atom from the gravitational field of the central star per second is

$$g\left(\frac{r}{r_n}\right)^2 m,$$

where g is the gravitational acceleration on the surface of the central star. However, not only the normal hydrogen atoms but also the free protons are subject to gravitational force. Therefore, the ratio μ of repulsive force R to attractive force G is given by

$$\mu = \frac{R}{G} = \frac{\kappa\,\pi}{mg\left(1 + \frac{n^+}{n_1}\right)} \frac{2h\nu_\alpha}{c^3} \int_{\nu_0}^{\infty} \frac{\nu^2 \, d\nu}{\exp\left(\frac{h\nu}{kT}\right) - 1},$$

where n^+ is the number of protons per cubic centimeter, n_1 is the number of normal hydrogen atoms in the same volume. Even in the case when we put

$$\frac{n^+}{n_1} = 500$$

which value is probably too high (see [5]), we obtain for $T = 40,000°$.

$$\mu = \frac{10^{10}}{g}.$$

The value of g for the nuclei of planetary nebulae will be much larger than that for the Sun. But it is very improbable that it may reach 10^{10} cm sec^{-2}. Therefore, we may conclude that if the optical thickness of the nebula in the ultraviolet region is not too small, the radiation pressure will be the dominant factor in the exterior parts of the nebula.

Remark. The essential point was the study of the $\mathbf{L}_\alpha$-field. The second part of the Pulkovo paper was devoted to helium nebulae. It is omitted here because since its appearance 60 years ago more complicated problems have been discussed and solved by other authors (the most important steps have been made by V. V. Sobolev).

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